

STRUCTURALISM



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play of anticipation and correction (feedback). As we shall see in Section 10, the range of application of feedback mechanisms is enormous.

Finally, there are regularities in the non-technical sense of the word which depend upon far simpler structural mechanisms, on rhythmic mechanisms such as pervade biology and human life at every level.⁷ Rhythm too is self-regulating, by virtue of symmetries and repetitions. Though the self-regulation that is here involved is of a much more elementary sort, it would not do to exclude rhythmic systems from the domain of structure.

Rhythm, regulation, operation—these are the three basic mechanisms of self-regulation and self-maintenance. One may, if one so desires, view them as the “real” stages of a structure’s “construction,” or, reversing the sequence, one may use operational mechanisms of a quasi-Platonic and non-temporal sort as a “basis” from which the others are then in some manner “derived.”

⁷ The study of such biological rhythms and periodicities (i.e., cycles of approximately 24 hours, which are remarkably general) has in recent years been turned into an entire new discipline with its own specialized mathematical and experimental techniques.



II

MATHEMATICAL AND LOGICAL STRUCTURES

5. Groups

A critical account of structuralism must begin with a consideration of mathematical structures, not only for logical but even for historical reasons. True, when structuralism first made its appearance in linguistics and psychology, the formative influences were not directly mathematical—Saussure’s concept of synchronic equilibrium was inspired by ideas then current in economic theory, and the Gestalt psychologists took off from physics. But the structural models of Lévi-Strauss, the acknowledged master of present-day social and cultural anthropology, are a direct adaptation of general algebra. Moreover, if we accept the general definition of structure sketched in the preceding chapter, the first known “structure,” and the first to be studied as such, was surely the

mathematical "group," Galois' discovery, which little by little conquered the whole of nineteenth-century mathematics.¹

A mathematical group is a system consisting of a set of elements (e.g., the integers, positive and negative) together with an operation or rule of combination (e.g., addition) and having the following properties:

1. performed upon elements of the set, the combinatory operation yields only elements of the set;
2. the set contains a neuter or identity element (in the given case, 0) such that, when it is combined with any other element of the set, the latter is unaffected by the combinatory operation (in the given case, $n + 0 = n$ and, since addition is commutative, $n + 0 = 0 + n = n$);
3. the combinatory operation has an inverse in the system (here subtraction) such that, in combination with the former, the latter yields the neuter or identity element ($+n - n = 0$);
4. the combinatory operation (and its inverse) is associative ($[n + m] + 1 = n + [m + 1]$).

Groups are today the foundation of algebra. The range and fruitfulness of the notion are extraordinary. We run into it in practically every area of mathematics and logic. It is already being used in an important way in physics, and very likely the day will come when it acquires a

¹ For an informal account of group theory, the reader might turn to Part IX, "The Supreme Art of Abstraction: Group Theory," in volume III of James R. Newman's *World of Mathematics* (New York: Simon and Schuster, 1956); for a somewhat more formal treatment, see Raymond L. Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley, 1960), Chapter VII. [Trans.]

central role in biology as well. Clearly, then, we should try to understand the reasons for the immense success of the group concept. Since groups may be viewed as a kind of prototype of structures in general, and since they are defined and used in a domain where every assertion is subject to demonstration, we must look to them to ground our hope for the future of structuralism.

The primary reason for the success of the group concept is the peculiar—mathematical or logical—form of abstraction by which it is obtained; an account of its formation goes far to explain the group concept's wide range of applicability. When a property is arrived at by abstraction in the ordinary sense of the word, "drawn out" from things which have the property, it does, of course, tell us something about these things, but the more general the property, the thinner and less useful it usually is. Now the group concept or property is obtained, not by this sort of abstraction, but by a mode of thought characteristic of modern mathematics and logic—"reflective abstraction"—which does not derive properties from *things* but from our ways of *acting on things*, the operations we perform on them; perhaps, rather, from the various fundamental ways of *coordinating* such acts or operations—"uniting," "ordering," "placing in one-one correspondence," and so on. Thus, when we analyze the concept of groups, we come upon the following very general coordinations among operations:

1. the condition that a "return to the starting point" always be possible (via the "inverse operation");
2. the condition that the same "goal" or "terminus" be attainable by alternate routes and without the itin-

erary's affecting the point of arrival (associativity").²

Because of these two restrictive conditions, group structure makes for a certain coherence—whatever has that structure is governed by an internal logic, is a self-regulating system. This self-regulation is really the continual application of three of the basic principles of rationalism: the principle of non-contradiction, which is incarnate in the reversibility of transformations; the principle of identity, which is guaranteed by the permanence of the identity element; and the principle, less frequently cited but just as fundamental, according to which the end result is independent of the route taken. To illustrate the last point, consider the set of displacements in space. It constitutes a group (since any two successive displacements yield a displacement, a given displacement can always be "annulled" by an inverse displacement or "return," etc.). That the associativity of the group of spatial displacements (equivalent to our intuitive notion of using a detour) is absolutely essential is seen as soon as it is recognized that, if termini did vary with the paths traversed to reach them, space would lose its coherence and thereby be annihilated; what we would have instead would be some sort of perpetual Heraclitean flux.

Group structure and transformation go together. But when we speak of transformation, we mean an intelligible change, which does not transform things beyond recognition at one stroke, and which always preserves invariance in certain respects. To return to our example, the

² If the group operation and its inverse are commutative, the group is commutative or "abelian"; otherwise it is non-commutative.

displacement of a solid in ordinary space leaves its dimensions unchanged; similarly, the sum of the parts of a whole remains invariant under continual division. The existence of groups reveals how contrived the antithesis of self-sameness and change, on which Meyerson based his entire epistemology, really is; whereas according to him identity alone is rational, all change irrational, the consistency of the group concept, which calls for a certain inseparable connection of identity and change, is proof of their compatibility.

It is because the group concept combines transformation and conservation that it has become the basic constructivist tool. Groups are systems of transformations; but more important, groups are so defined that transformation can, so to say, be administered in small doses, for any group can be divided into subgroups and the avenues of approach from any one to any other can be marked out. Thus, starting with the group of which we spoke just now, the group of displacements, which leaves not only the dimensions of the displaced body or figure invariant, but preserves its angles, parallels, straight lines, and so on, as well, we can go on to the next "higher" group by letting the dimensions vary while preserving the other properties enumerated. In this way we obtain the group of similar figures of bodies: shape is kept invariant under transformation of dimensions. The group of displacements has thereby become a subgroup of the shape group. Next we may allow the angles to vary while conserving parallels and straight lines. A still more general group, that treated by "affine geometry" (which deals with such problems as how to transform one lozenge into

another), is thus obtained, of which the shape group now becomes a subgroup. Continuing this process, parallels may be modified while straight lines are preserved; the "projective" group is thereby constructed; and the entire preceding series now becomes a "stack" of subgroups within the projective group. Finally, even straight lines may be subjected to transformation. Shapes are now treated as if they were elastic: only "biunique" and "bi-continuous" correspondence among their points are preserved under transformation. The group thus obtained (that of "homeomorphs") is the most general. It constitutes the subject matter of topology. The various kinds of geometry³—once taken to be static, purely representational, and disconnected from one another—are thus reduced to one vast construction whose transformations under a graded series of conditions of invariance yield a "nest" of subgroups within subgroups. It is this radical change of the traditional representational geometry into one integrated system of transformations which constitutes Felix Klein's famous Erlanger Program. The Erlanger Program⁴ is a prime example of the scientific fruitfulness of structuralism.

³ The various metric geometries—Euclidean and non-Euclidean—can be constructed by applying a "general metrics" to topology; that is, it is possible to "reverse directions" and to descend from the group of maximum generality all the way down to the group of displacements with which our account started.

⁴ See Felix Klein, *Elementary Mathematics from an Advanced Standpoint: Geometry*, trans. E. R. Hedrick and C. A. Noble (New York: Dover, 1939). [Trans.]

6. "Parent Structures"

But in the eyes of contemporary structuralist mathematicians, like the Bourbaki,⁵ the Erlanger Program amounts to only a partial victory for structuralism, since they want to subordinate all mathematics, not just geometry, to the idea of structure.

Classical mathematics is a quite heterogeneous collection of algebra, theory of numbers, analysis, geometry, probability calculus, and so on. Each of these has its own delimited subject matter; that is, each is thought to deal with a certain "species" of objects. The mathematicians of the Bourbaki circle, having noted that sets of the most various sorts, not just algebraic sets, may display the group property, and intent upon overcoming the traditional compartmentalization of mathematics into areas that exist simply side by side, initiated a program of generalization whereby the group structure becomes only one among a variety of basic structures. If the term "element" is applicable to objects as abstract as numbers, displacements, projection, and so on (some of which are, as we have seen, resultants of operations as well as operators), this means that group structure is quite independent of the intrinsic nature of its elements, which can, accordingly, be left unspecified. Transformations may be disengaged from the objects subject to such transformation and the group defined solely in terms of the set of trans-

⁵ "Nicolas Bourbaki" is the collective pseudonym of a group of French mathematicians who publish under that name. [Trans.]

formations. The Bourbaki program consists essentially in extending this procedure by subjecting mathematical elements of every variety, regardless of the standard mathematical domain to which they belong, to this sort of "reflective abstraction" so as to arrive at structures of maximum generality.

It is worth noting that in its initial stages execution of the Bourbaki program called for some sort of "induction," since neither the form of the basic structures sought nor their number was known *a priori*. This quasi-inductive procedure led to the discovery of three "parent structures," that is, three not further reducible "sources" of all other structures.⁶ The first of these is the *algebraic* structure. The prototype of this family is the mathematical group, together with all its derivatives—rings, fields, and so on. The characteristic of structures belonging to the algebraic family is that "reversibility" takes the form of "inversion" or "negation" (if T is the operator and T^{-1} its inverse, $T \cdot T^{-1} = 0$ in all algebraic structures). Next there are *order* structures. The "lattice" or "network" is their prototype, a structure as general as the group structure, but one which was not studied until comparatively recently (by Birkhoff, Glivenko, and others). Networks unite their elements by the predecessor/successor relation, any two elements of the network having a smallest upper bound (the nearest of the successors, or *supremum*) and a greatest lower bound (the closest predecessor, or *infimum*). Like the group property, the lattice property has a very wide range of application: it

⁶ The number 3 may cause suspicion, so let us repeat—that there are just three such principal structures was a discovery, the outcome of "regressive analysis," not a postulate.

applies, for example, to the set of subsets of a collection, or to a group of its subgroups.⁷ The defining mark of order structures is that reversibility takes the form, not of inversion, as in groups, but of reciprocity: " $(A \cdot B)$ precedes $(A + B)$ " transforms into " $(A + B)$ succeeds $(A \cdot B)$ " by permutation of the "+" and "•" operators and the predecessor and successor relations. Finally, there are the topological structures of which we spoke in the preceding section: neighborhood, continuity, and limit are here the basic conceptions.

Once these three parent structures have been distinguished and characterized, the rest follow, that is, can be constructed. There are two methods of construction, combination and differentiation: a set of elements may be subjected to the restrictive conditions of two parent structures at the same time (algebraic topology, for instance, is yielded by combining algebraic and topological conditions); or substructures of any one of the three parent structures may be defined by introducing certain additional restrictive conditions. (This is the procedure we saw at work in Section 5 above.) The reverse procedure, "de-differentiation," is also possible: one may drop a restrictive condition and thereby move from a "stronger" to a "weaker" structure: the semigroup, for instance, may be defined as the structure resulting from the deletion of conditions 2 and 4 in the group definition given on page 18 above; the natural numbers greater than 0 constitute such a semigroup.

⁷ A set S of n elements has 2^n subsets, since the set of subsets is obtained by taking the elements one by one, two by two, and so on, and the null set as well as the set S itself are counted as subsets.

Does this "mathematical architecture"⁸ build on foundations that are in some manner "natural," or are the Bourbaki parent structures simply an axiomatic basis of their system? The question is not only interesting in its own right but will help us tie things together and clarify the general significance of structures. If we take the adjective in roughly the sense in which we speak of the positive whole numbers as "natural," the Bourbaki parent structures do appear to be natural. The positive whole numbers antedate mathematics; they are constructed by means of operations that stem from ordinary, everyday activities such as the "matching" to which even very primitive societies resort in their barter transactions, or at which we catch the playing child. When Cantor defined the first cardinal infinite in terms of one-one correspondence, he utilized an operation which, in its "natural" form, precedes nineteenth-century mathematics by uncounted millennia. Now when we study the intellectual development of the child, we find that the earliest cognitive operations, those which grow directly out of handling things, can be divided into precisely three large categories, according to whether reversibility takes the form of "inversion," of "reciprocity," or of "continuity" and "separation." Corresponding to the first—formally considered, algebraic structures—there are classificatory and number structures; corresponding to the second—formally considered, order structures—there are series and serial correspondences; corresponding to the last—formally considered, topological structures—there are operations that yield classes, not in terms of resemblances

⁸ We borrow the expression from Bourbaki literature.

and differences, but in terms of "neighborhoods," "continuity," and "boundaries." It is remarkable that, psychogenetically, topological structures antedate metric and projective structures, that psychogenesis inverts the historical development of geometry but matches the Bourbaki "genealogy"!

These facts seem to suggest that the mother structures of the Bourbaki correspond to coordinations that are necessary to all intellectual activity, though they be very elementary, even rudimentary, and quite lacking in generality in the earliest stages of intellectual development. It would, in fact, not be difficult to show that in these very early stages intellectual operations grow directly out of sensory-motor coordinations, and that *intentional* sensory-motor acts—the human baby's or the chimpanzee's—*cannot* be understood apart from "structures" (see Chapter IV).

Before we sketch the implications of the foregoing observations for logic, we want to call the reader's attention to the fact that the structuralism of the Bourbaki circle is in process of transformation, under the influence of a current of thought well worth noting because it shows how new structures are, if not "formed," at least "discovered." What we have in mind are the "categories" of MacLane, Eilenberg, and others. The "categories" of the new branch of the Bourbaki school are classes which comprise "functions" and therefore "morphisms" among their elements (a function, in the usual acceptation of the word, being the "application" of one set to another [or itself], obviously engenders isomorphisms, in fact, every variety of "morphism"). Suffice it to say here that the "categories," with their emphasis upon functions, no

longer revolve around parent structures but around the acts of correlation by which the latter were obtained: these new structures are not built up by in some way compounding the beings yielded by certain antecedent operations but by correlating these very operations, the formational procedures themselves.

There is some justice, then, in S. Papert's observation that MacLane's categories are a device for laying hold of mathematical operations rather than of "mathematics itself": they constitute yet another example of that reflective abstraction which derives its substance, not from objects, but from operations performed upon objects, even when the latter are themselves products of reflective abstraction.

These facts have an important bearing both on the nature and on the manner of construction of structures.

7. *Logical Structures*

Since logic is concerned with the form of knowledge, not its content, it is *prima facie* a privileged domain for structures. Yet, as we hinted earlier, this is not so if by "logic" we mean "mathematical" or "symbolic" logic, the only logic that really counts today. On the other hand, if we start from a rather broader perspective, such as allows us to raise the problem of "natural logic" (in approximately the sense in which we spoke of "natural numbers" in Section 6), we find that the "opposites" in terms of which we just now characterized logic, namely, "form" and "content," are correlatives, not absolutes: the "contents" on which logical forms are imposed are not

formless; they have forms of their own; else they could not "potentially be logicized." And the forms of what originally appeared to be "pure content" in turn themselves *have* content, though less distinctly made out, a content with its own form, and so on, indefinitely, each element being "content" relative to some prior element and "form" for some posterior element.

For structuralist theory this "nesting" of forms and the correlativity of form and content are of the utmost significance, but from the logical point of view these facts are of no interest, unless indirectly, through their bearing on the problem of the limits of formalization.

Symbolic logic proceeds as follows: it assumes some arbitrary position in the ascending contents/form series and turns its starting point into an absolute beginning—into the "basis" of a "logical system." More explicitly, the basis of such an axiomatic system consists of (1) certain primitive or undefined conceptions, which serve to define the rest; (2) certain axioms or undemonstrated propositions, which serve to demonstrate the rest. The undefined conceptions are primitive or *indefinable* and the undemonstrated propositions axiomatic or *indemonstrable* within the particular system under consideration, but they may well turn up derived in some other system. And the axiomatic method leaves the logician free to choose what system he pleases; all it requires is that the primitive conceptions and the propositions which serve as axioms be "adequate," compatible, and "mutually independent" (that is, reduced to a minimum). Next, there will have to be certain rules of construction, i.e., formation and transformation procedures. The formal system thus obtained is self-sufficient, dispenses with intuition, and is,